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VARIATIONAL APPROACH FOR APPROXIMATE ANALYTICAL SOLUTION TO NON-NATURAL VIBRATION EQUATIONS*

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Abstract– In this study, He's Variational Approach (VA) is used to solve the non-natural vibrations and oscillations. The method works very well for the whole range of initial amplitudes and does not demand small perturbation. It is also sufficiently accurate to both linear and nonlinear physics and engineering problems. We consider some examples to illustrate the effectiveness and convenience of the method. Runge-Kutta's [RK] algorithm was also implemented to show the examples through a numerical method. Finally, to show the accuracy of the VA, the results have been shown graphically and compared with numerical and exact solution.

Keywords– Variational approach (VA), nonlinear oscillators, analytical method

1. INTRODUCTION

One of the most interesting areas in many physics and engineering problems is nonlinear vibrations. It is very important in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of their motions. Recently, many researchers have been working on the analytical and numerical methods in nonlinear vibrations such as: homotopy perturbation method [1-2], energy balance method [3], variational iteration method [4], amplitude frequency formulation [5-8], max-min approach [9-10], Hamiltonian approach [11], variational approach [21-14], homotopy analysis method[15], and the other analytical and numerical methods [16-35]. Among these methods, variational approach is considered to solve the nonlinear vibration equations in this paper.

The paper is organized as follows:

First, the basic concept of He's variational approach and Runge-Kutta's algorithm are described. Then, the applications of He's variational approach have been studied to demonstrate the applicability and preciseness of the method for two examples. Some comparisons between analytical and numerical solutions are presented. Eventually it is shown that VA can converge to a precise cyclic solution for nonlinear systems.

2. BASIC CONCEPT OF VARIATIONAL APPROACH (VA)

He suggested a variational approach which is different from the known variational methods in open literature [12]. Hereby we give a brief introduction of the method:

$$\ddot{u} + f\ (u) = 0 \tag{1}$$

Its variational principle can be easily established utilizing the semi-inverse method [12];

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}\dot{u}^2 + F(u) \right) dt \tag{2}$$

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Where T is period of the nonlinear oscillator, $\partial F / \partial u = f$. Assume that its solution can be expressed as

$$u(t) = A\cos(\omega t) \tag{3}$$

Where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (3) into Eq. (2) results in:

$$\begin{aligned} J(A,\omega) &= \int_0^{T/4}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t + F(A\cos\omega t)\right)dt \\ &= \frac{1}{\omega}\int_0^{\pi/2}\left(-\frac{1}{2}A^2\omega^2\sin^2 t + F(A\cos t)\right)dt \\ &= -\frac{1}{2}A^2\omega\int_0^{\pi/2}\sin^2 t\ dt + \frac{1}{\omega}\int_0^{\pi/2}F(A\cos t)dt \end{aligned} \tag{4}$$

Applying the Ritz method, we require:

$$\frac{\partial J}{\partial A} = 0 \tag{5}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{6}$$

But with a careful inspection, for most cases, He found that

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2}A^2\int_0^{\pi/2}\sin^2 t\,dt - \frac{1}{\omega^2}\int_0^{\pi/2}F(A\cos t)dt < 0 \tag{7}$$

Thus, He modified conditions Eq. (5) and Eq. (6) into a simpler form:

$$\frac{\partial J}{\partial \omega} = 0 \tag{8}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

3. BASIC IDEA OF RUNGE-KUTTA'S METHOD (RK)

For the numerical approach to verify the analytic solution, the fourth RK (Runge-Kutta) method has been used. This iterative algorithm is written in the form of the following formulae for the second-order differential equation:

$$\begin{aligned} \dot{u}_{i+1} &= \dot{u}_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + h_4\right) \\ u_{i+1} &= u_i + \Delta t\left(\dot{u}_i + \frac{\Delta t}{6}\left(h_1 + h_2 + h_3\right)\right) \end{aligned} \tag{9}$$

where, Δt is the increment of the time and h_1, h_2, h_3, and h_4 are determined from the following formula:

$$\begin{aligned} h_1 &= f\left(\dot{u}, u_i, \dot{u}_i\right), \\ h_2 &= f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i,\ \dot{u}_i + \frac{\Delta t}{2}h_1\right), \\ h_3 &= f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i,\ \frac{1}{4}\Delta t^2 h_1,\ \dot{u}_i + \frac{\Delta t}{2}h_2\right), \\ h_4 &= f\left(t_i + \Delta t,\ u_i + \Delta t\,\dot{u}_i,\ \frac{1}{2}\Delta t^2 h_2,\ \dot{u}_i + \Delta t\,h_3\right). \end{aligned} \tag{10}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative are determined from initial condition. Then, with a small time increment Δt , the displacement function and its first-order derivative at the new position can be obtained using Eq. (9). This process continues to the end of the time limit.

4. APPLICATION

In order to assess the advantages and the accuracy of the variational approach, the following examples are considered:

a) Example 1

An example of a single degree of freedom conservative system has been considered that is described by an equation as follows. A rigid rod is rigidly attached to the axle as shown in Fig. 1. The wheels roll without slip as the pendulum swings back and forth. The wheel is restrained by a spring which is fixed to a wall on the other side. Only the ball on the end of the pendulum has appreciable mass and it may be considered as a particle. The governing equation of the motion is [23]:

$$m\left(l^2+r^2-2rl\cos(\theta)\right)\ddot{\theta}+mrl\sin(\theta)\dot{\theta}^2+mgl\sin(\theta)+kr^2\theta=0 \tag{11}$$

with initial conditions;

$$\theta(0)=A, \qquad \dot{\theta}(0)=0. \tag{12}$$

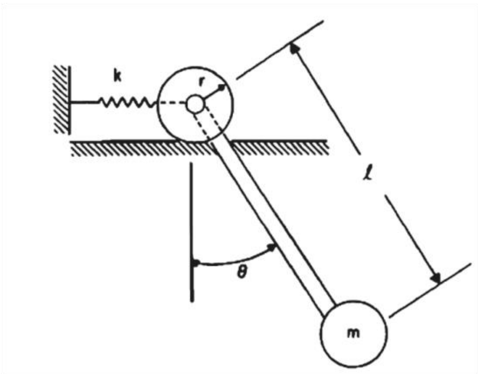

Fig. 1. Pendulum attached to rolling wheels that are restrained by a spring [23]

In order to apply the variational approach method to solve the above problem, the approximations $\cos\theta\approx 1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4$ and $\sin\theta\approx\theta-\frac{1}{6}\theta^3$ are used.

The variational formulation can be readily obtained from Eq. (11) as follows:

$$J(\theta)=\int_0^t\begin{pmatrix}\frac{1}{2}ml^2\dot{\theta}^2+\frac{1}{2}mr^2\dot{\theta}^2-mrl\,\dot{\theta}^2-\frac{1}{24}mgl\,\theta^4\dot{\theta}^2+\\ \frac{1}{2}mrl\,\dot{\theta}^2\theta^2-\frac{1}{24}mgl\,\theta^4+\frac{1}{2}kr^2\theta^2\end{pmatrix}dt\,. \tag{13}$$

Choosing the trial function as $\theta(t)=A\cos(\omega t)$ in Eq. (13), we obtain:

$$J(A)=\int_0^{T/4}\begin{pmatrix}\frac{1}{2}ml^2A^2\omega^2\sin^2(\omega t)+\frac{1}{2}mr^2A^2\omega^2\sin^2(\omega t)-mrlA^2\omega^2\sin^2(\omega t)\\ -\frac{1}{24}mrlA^6\omega^2\cos^4(\omega t)\sin^2(\omega t)+\frac{1}{2}mrlA^4\omega^2\sin^2(\omega t)\cos^2(\omega t)\\ +\frac{1}{2}mglA^2\cos^2(\omega t)-\frac{1}{24}mglA^4\cos^4(\omega t)+\frac{1}{2}kr^2A^2\cos^2(\omega t)\end{pmatrix}dt \tag{14}$$

The stationary condition with respect to A leads to:

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\begin{pmatrix} ml^2A\omega^2\sin^2(\omega t)+mr^2A\omega^2\sin^2(\omega t)-2mrlA\,\omega^2\sin^2(\omega t) \\ -\frac{1}{4}mrlA^5\omega^2\cos^4(\omega t)\sin^2(\omega t)+2mrlA^3\omega^2\sin^2(\omega t)\cos^2(\omega t) \\ +mgl\,A\cos^2(\omega t)-\frac{1}{6}mglA^3\cos^4(\omega t)+kr^2A\cos^2(\omega t) \end{pmatrix}dt=0 \tag{15}$$

Solving Eq. (15), according to ω, we have

$$\omega^2=\frac{\int_0^{\frac{\pi}{2}} mglA\cos^2 t-\frac{1}{6}mglA^3\cos^4 t}{\int_0^{\frac{\pi}{2}}\begin{pmatrix} ml^2A\sin^2 t+2mrlA^3\sin^2 t\cos t+mrA^2\sin^2 t \\ -2mrlA\sin^2 t-\frac{1}{4}mrlA^5\sin^2 t\cos^4 t\end{pmatrix}} \tag{16}$$

Then we have

$$\omega_{VA}=2\sqrt{\frac{mglA^3-8kr^2-Amgl}{m\left(32r^2+A^4rl+64ml-32l^2-16A^2rl\right)}} \tag{17}$$

According to $\theta(t)=A\cos(\omega t)$ and Eq. (17), we can obtain the following approximate solution:

$$\theta(t)=A\cos\left(2\sqrt{\frac{mglA^3-8kr^2-Amgl}{m\left(32r^2+A^4rl+64ml-32l^2-16A^2rl\right)}}\,t\right) \tag{18}$$

b) Example 2

The motion of a particle on a rotating parabola. The governing equation of motion and initial conditions can be expressed as:

$$(1+4q^2u^2)\ddot{u}+4q^2u\dot{u}^2+\Delta u=0 \qquad u(0)=A,\quad \dot{u}(0)=0 \tag{19}$$

where $q>0$ and $\Delta>0$ are known positive constants.

Variational formulation of Eq. (19) can be readily obtained as follows:

$$J(u)=\int_0^t\left(\frac{1}{2}\dot{u}^2+2q^2u^2\dot{u}^2+\frac{1}{2}\Delta u^2\right)dt \tag{20}$$

Substituting the trial function $u(t)=A\cos(\omega t)$ into Eq. (20), we obtain**:**

$$J(A)=\int_0^{T/4}\left(\frac{1}{2}A^2\omega^2 sin^2(\omega t)+2q^2A^4\omega^2 sin^2(\omega t)cos^2(\omega t)+\frac{1}{2}\Delta A^2cos^2(\omega t)\right)dt \tag{21}$$

The stationary condition with respect to A leads to:

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\left(A\,\omega^2 sin^2(\omega t)+8qA^3\omega^2 sin^2(\omega t)cos^2(\omega t)+\Delta A cos^2(\omega t)\right)dt=0 \tag{22}$$

or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\left(A\,\omega^2 sin^2 t+\Delta A cos^2 t+8qA^3\omega^2 sin^2 t\;cos^2 t\right)dt=0 \tag{23}$$

Solving Eq. (23), according to ω, we have

$$\omega^2 = \frac{\int_0^{\frac{\pi}{2}} \left(\Delta A cos^2 t\right) dt}{\int_0^{\frac{\pi}{2}} \left(A\, sin^2 t + 8q\, A^3 sin^2 t\; cos^2 t\right) dt} \tag{24}$$

Then we have

$$\omega_{VA} = \sqrt{\frac{\Delta}{1+2A^2 q^2}} \tag{25}$$

According to Eqs. (3) and (25), we can obtain the following approximate solution:

$$u(t) = A\cos\left(\sqrt{\frac{\Delta}{1+2A^2 q^2}}\, t\right) \tag{26}$$

The exact period is [31]:

$$\omega_{Exact} = 2\pi \Big/ 4A \int_0^{\pi/2} \frac{\sqrt{1+4q^2 A^2 \cos^2 t}\; sin\, t}{\sqrt{\Delta A^2 \sin^2 t}}\, dt \tag{27}$$

5. RESULTS AND DISCUSSIONS

To illustrate and verify the accuracy of this approximate analytical approach, some comparisons of the analytic responses with the numerical solutions and exact solutions are presented in Figs. 2 to 5 for example 1, and Table 1, and Figs. 6 to 9 for example 2.

In example 1, Fig. 2a represents the displacement time history and Fig. 2b is the phase curve. Figure 3 is the influence of axle length (l) and radius of wheel (r) on nonlinear frequency. Figure 4 is influence of sprig stiffness (k) of system on nonlinear frequency based on various parameters. Sensitive analyses on the nonlinear frequency are shown in Fig. 5.

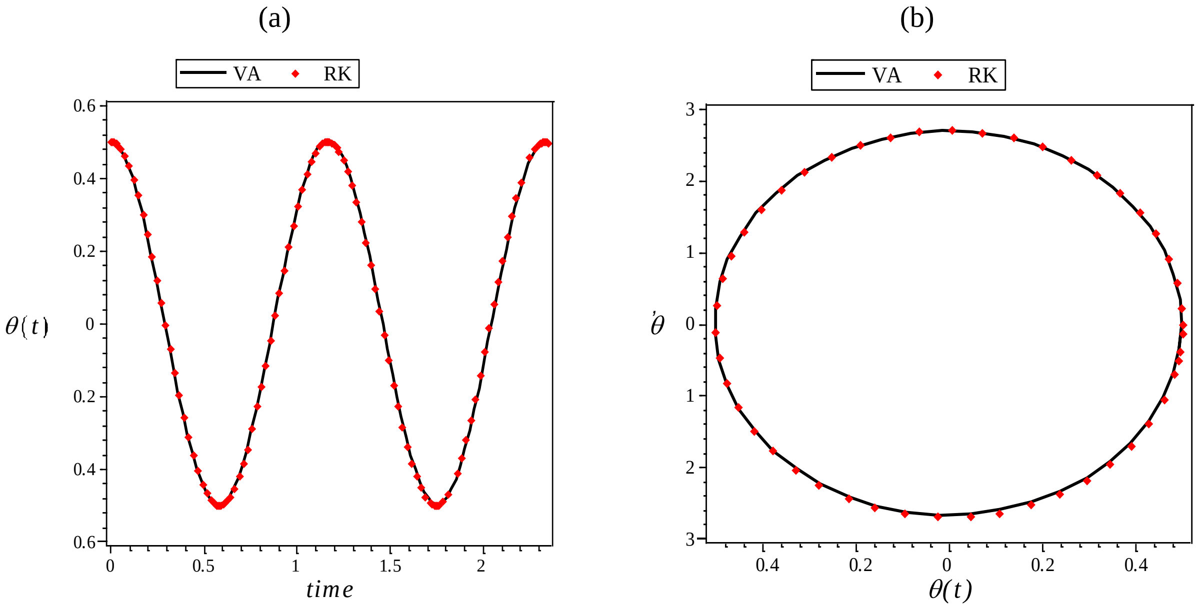

Fig. 2. Comparison of analytical solution of time history and phase curve with the numerical solution for $m=5,\quad l=1,\quad r=0.2,\quad g=9.81,\quad k=50,\quad A=2$

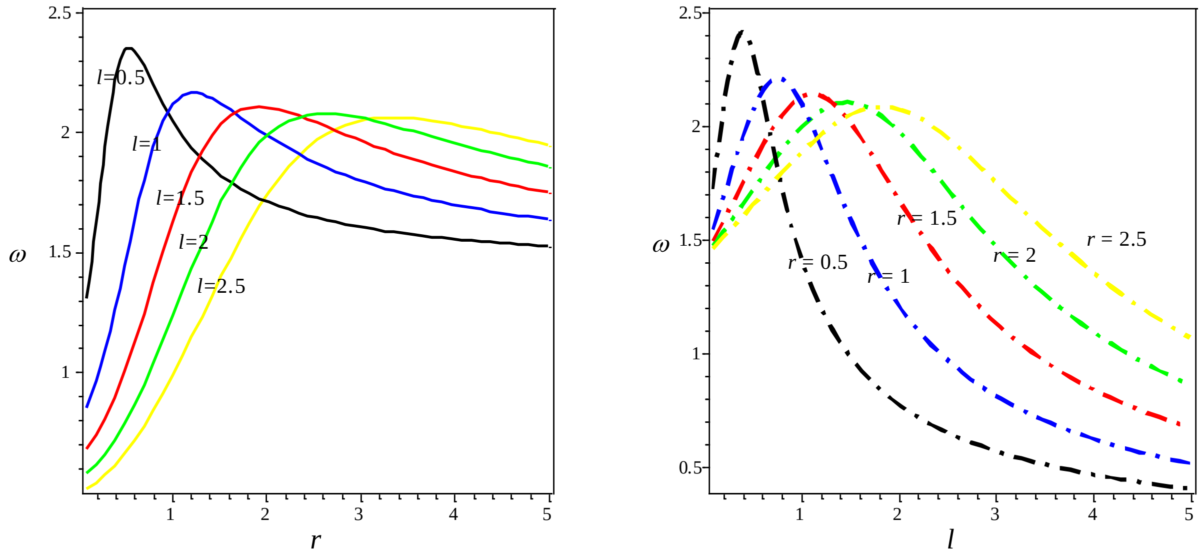

Fig. 3. Influence of axle length (l) and radius of wheel (r) on nonlinear frequency for
$m = 5, \quad g = 9.81, \quad k = 100, \quad A = 10$

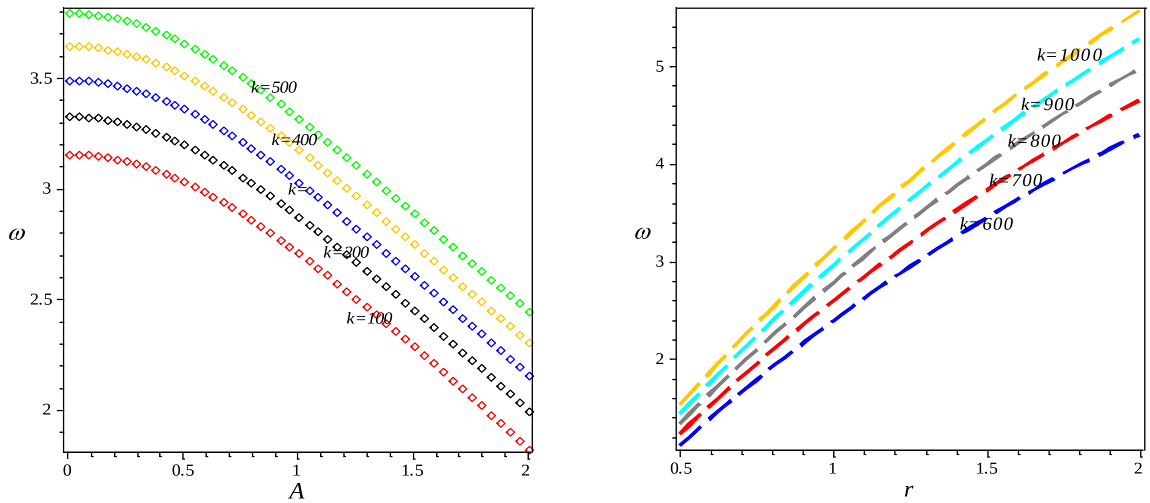

Fig. 4. Influence of spring stiffness (k) of system on nonlinear frequency base on various parameters

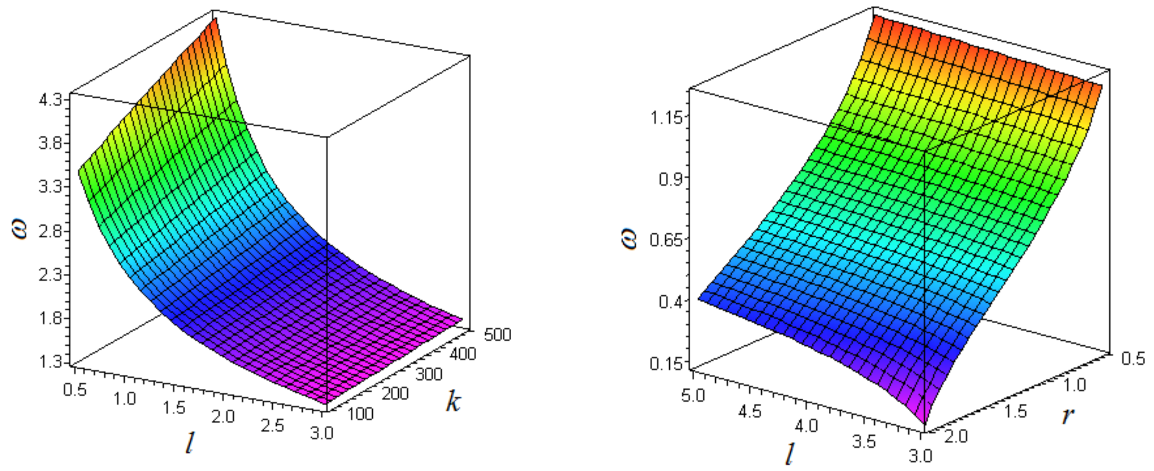

Fig. 5. Sensitivity analysis of various parameter of system on nonlinear frequency

In example 2, Table 1 presents the comparison of the obtained results with the exact solution for different values of A, q, Δ and the maximum relative error is less than 2.1399%. Figures 6a and 7a represent comparison of the analytical solution of $u(t)$ based on time with the exact solution. These figures show the behavior of the oscillation is periodic. Figures 6b and 7b are the phase plan curves ($\dot{u}(t)$ versus $u(t)$ curve) of the problem. The comparison of analytical solution based on time with the exact solution shows an excellent agreement of the applied method. Figure 8 is the effect of amplitude and Δ on the nonlinear frequency of the system. A sensitive analysis is also done on the nonlinear frequency of the system by considering the effect of A and Δ simultaneously.

Table 1. Comparison of the approximate and exact frequencies corresponding to various parameters in Eq. (25)

A	q	Δ	ω_{VA}	ω_{Exact}	Error%
0.5	1	0.5	0.5774	0.5815	0.7135
0.5	0.5	2	1.3333	1.3344	0.0774
1	0.8	1.5	0.8111	0.8288	2.1399
1	0.7	0.5	0.5025	0.5108	1.6300
1.5	0.5	2	0.9701	0.9888	1.8836
1.5	0.3	2.5	1.3339	1.3410	0.5298
2	0.2	4	1.7408	1.7473	0.3725
2	0.4	1	0.6623	0.6767	2.1399

(a) (b)

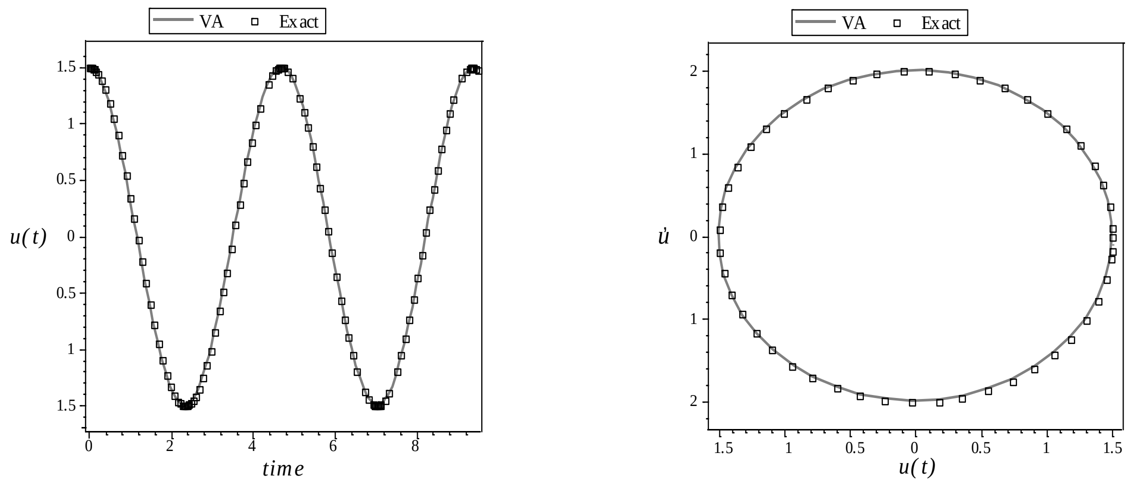

Fig. 6. Comparison of analytical solution of time history and phase curve with the exact solution for $A = 1.5, \quad q = 0.3, \quad \Delta = 2.5$

(a) (b)

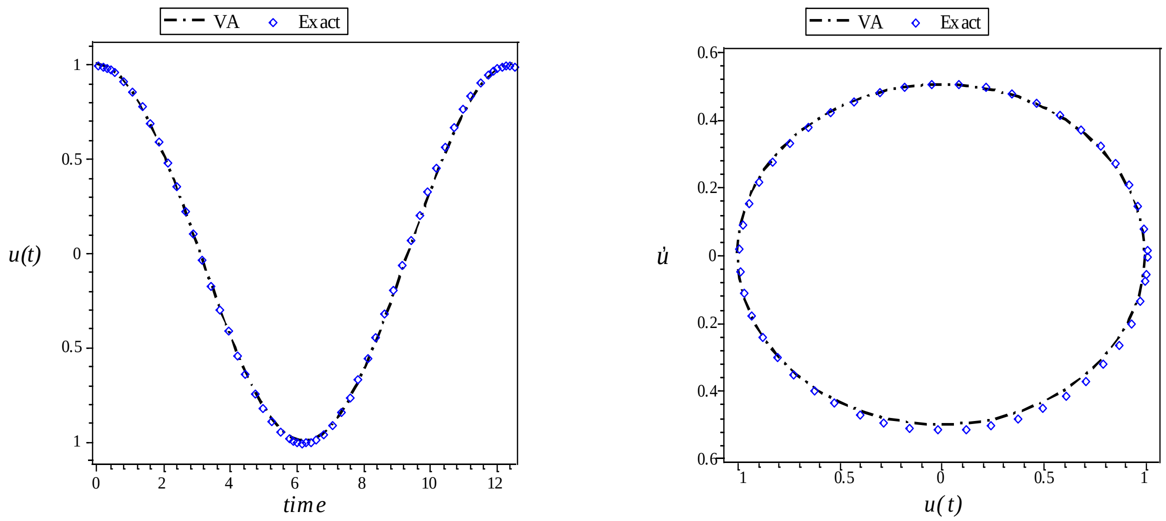

Fig. 7. Comparison of analytical solution of time history and phase curve with the exact solution for $A = 1, \quad q = 0.7, \quad \Delta = 0.5$

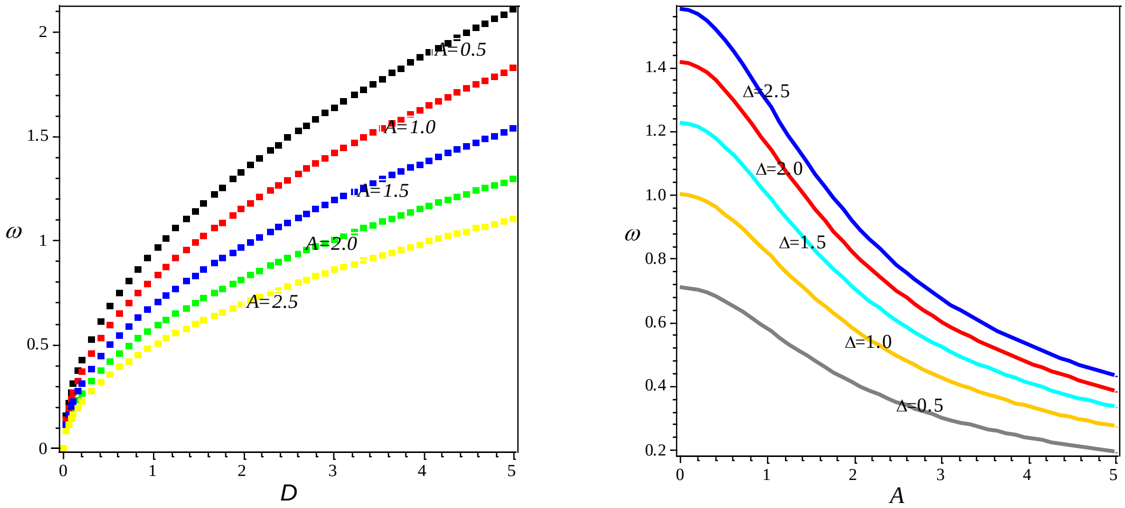

Fig. 8. Influence of various constant parameter of system on nonlinear frequency

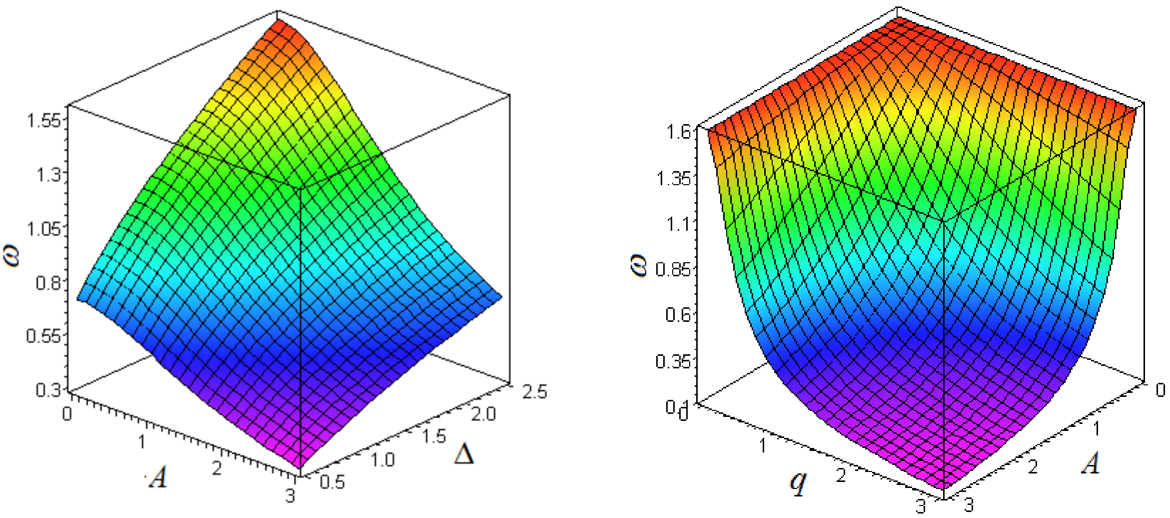

Fig. 9. Sensitivity analysis of various parameter of system on nonlinear frequency

It is evident that VA shows high accuracy with the numerical solution and is quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the VA could be potentiality used for the analysis of strongly nonlinear oscillation problems.

6. CONCLUSION

In this paper an uncomplicated but productive new method for non-natural oscillators called He's variational approach is used to solve high nonlinear oscillators. Two strong examples have been studied to show the accuracy and convergence of the method. It has been proven that the variational approach is very efficient, comfortable and sufficiently exact in engineering problems. Variational approach can be simply extended to any nonlinear equation for the analysis of nonlinear systems. The obtained results from the approximate analytical solutions are in excellent agreement with the corresponding numerical solutions.

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