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# ANALYTICAL APPROXIMATE SOLUTION FOR NONLINEAR VIBRATION OF MICROELECTROMECHANICAL SYSTEM USING HE'S FREQUENCY AMPLITUDE FORMULATION*

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**Abstract–** In this paper, nonlinear free vibration of a micro-beam-based micro electro mechanical oscillator system is studied using He's frequency amplitude method. Firstly, by considering mid-plane stretching effects and distributed electrostatic forces and implementing Euler-Bernoulli hypothesis, the governing equations are derived. Then, Galerkin method is used to convert the resulted partial differential equations into an ordinary differential one. At last this nonlinear ordinary differential equation is solved by He's frequency Amplitude formulation. The proposed method offers an analytical closed form solution which is accurate and simple and in order to study the problem parametrically and in a more general manner the variables are defined in normalized form. A comparison between the obtained results and the one by an accurate numerical method shows the good accuracy of this method in solving this kind of problem. In addition, the relationship between the nonlinear principal frequency and non-dimensional amplitude, electro-elastic and axial loads on the beams has been studied.

**Keywords–** Nonlinear vibration, microelectromechanical system, Euler-Bernoulli beam, He's frequency amplitude formulation

## 1. INTRODUCTION

Micro-electro-mechanical systems are small integrated devices that combine electrical and mechanical components. MEMSs are being increasingly used in a broad field of modern technology. Their Geometric simplicity, low thermal conductivity, low power consumption and specific weight have made them a preferred choice over traditional mechanical systems. In addition, they can easily be integrated into large layers and are highly resistant to vibrations and shocks [1].

The application of MEMS devices, especially those devices which require few mechanical components and small voltage levels for actuation are continuously growing. By means of Micro-electro-mechanical systems, it is now easy to fabricate mechanical elements such as beams, gears, diaphragms and springs for integrated circuits. These circuits benefit from the large surface-to-volume ratio of MEMSs that are produced with lower costs and higher reliability [2].

Micromechanical resonators or micro resonators are limited to functional amplitude of vibration. This limit may be set by material strength, or as is the case for MEMS resonators, unwanted nonlinear effects. In addition, electrostatic actuation, large deflections and damping caused by different sources amplify nonlinear behavior of MEMSs. For example, the micro-resonator linear resonance responses in the presence of a nonlinear restoring force shifts to a nonlinear one as the vibration amplitude increases. This phenomenon is called the Duffing effect. The amplitude-frequency curve shows hysteresis at sufficiently

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large amplitudes; moreover, it has been observed that vibration amplitudes depend on the direction of frequency sweep. Taking these things into consideration and the fact that the nonlinear oscillation of the resonator affects the performance of the resonator itself and other MEMS devices, study of nonlinear oscillation characteristics of MEMSs seems to be of a great importance.

While obtaining an exact analytical solution for nonlinear vibration of MEMS components, due to the complexity of the equation of motion has been found to be particularly difficult, several attempts have been made to find numerical and approximate analytical solutions. The shooting method [3], the differential quadrature method [4] and the integral equation method [5] are some of the numerical approaches. In recent years, application of the approximate analytical methods in solving such problems has grown widely because of their simplicity and high accuracy. One of the main approximate analytical approaches on nonlinear vibration analysis is Perturbation technique [6].

Traditional perturbation methods have many shortcomings, for example they are valid just for weakly nonlinear problems or cannot be applied directly if there is no small parameter in the equation. In order to overcome these shortcomings, many asymptotic techniques have been developed to obtain periodic solutions for problems with strong nonlinearity. Some of them are Variational Iteration Method (VIM) [7], Energy Balance Method (EBM) [8], Homotopy Perturbation method [9], Lindstedt–Poincaré method [10] and harmonic balance method [11].

In this paper, nonlinear MEMS electro-statically actuated oscillator is modeled as an Euler-Bernoulli beam. The effects of mid-plane stretching and Von-Karman geometric nonlinearity are also considered. The boundary conditions at both ends are assumed to be clamped-clamped and the oscillator is excited by an electric voltage. The governing dynamics are converted to non-dimensional equations in order to study the problem parametrically. He's amplitude frequency formulation [12-14] is used to solve the equation. The resulted closed form solution would allow us to investigate effect of different parameters on nonlinear behavior of the system such as the non-dimensional initial deformation amplitude and the non-dimensional exciting voltage. An evaluation has been done according to the results obtained from the energy balance method (EBM) [6] as an analytical method and the fourth-order Runge-Kutta method as a high accuracy numerical method.

## 2. PROBLEM FORMULATIONS

Consider a microbeam of length $l$, width $h$, and thickness $b$, initial gap $g_0$ and electrostatic load $V$ as shown in Fig. 1. The beam is located between two stationary electrodes and the boundary condition is supposed to be clamped-clamped. The electric load consists of two parts: a DC polarization voltage and an AC voltage. The DC component by applying an electrostatic force on the microbeam, deflects it to a new equilibrium position and the AC component vibrates the microbeam around this new equilibrium position. The electric load and the mechanical restoring force characterize the behavior of the microbeam [15]. The area and moment of inertia of the microbeam cross-section are $S=bh$ and $I=bh^3/12$, respectively.

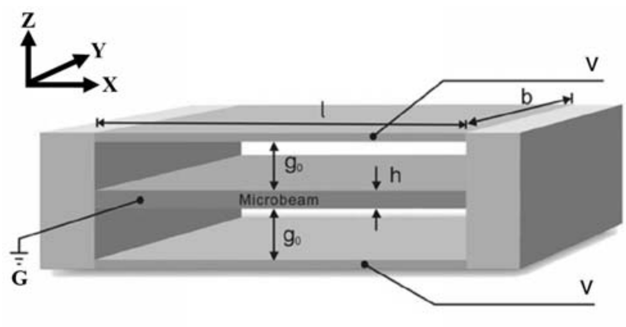


Fig. 1. The microbeam-based electromechanical resonator which is clamped between two stationary electrodes [2]

The nonlinear partial differential equation of the transverse motion considering the effect of mid-plane stretching is considered as [16]:

$$\bar{E}I\,\frac{\partial^4 w}{\partial x^4}+\rho S\,\frac{\partial^2 w}{\partial t^2}=\left[\bar{N}+\frac{\bar{E}S}{2l}\int_0^l(\frac{\partial w}{\partial x})^2\right]\frac{\partial^2 w}{\partial x^2}+q(x,t) \tag{1}$$

where $w(x,t)$ is the transverse deflection, $E$ is the Young's modulus, $\upsilon$ is the Poisson's ratio and $\bar{E}$ is the effective modulus of the microbeam. The value of $\bar{E}$ varies with different thicknesses of the microbeam as follows:

$$\bar{E}=\begin{cases}\dfrac{E}{1-\upsilon^2} & \text{The Wide Microbeam (b} \geq \text{5h)}\\ E & \text{The Narrow Microbeam (b} < \text{5h)}\end{cases}$$

$\bar{N}$ denotes the tensile or compressive axial load and is dependent on the mismatch of both thermal expansion coefficient and crystal lattice period between substrate and the microbeam. The $q(x,t)$ is the driving force per unit length resulted by electrostatic excitation and is computed as [17]:

$$q(x,t)=\frac{\varepsilon_v b v^2}{2}\left[\frac{1}{(g_0-w(x,t))^2}-\frac{1}{(g_0+w(x,t))^2}\right] \tag{2}$$

where $\varepsilon_v=8.85\ pF/m$ is the dielectric constant of the gap medium.
The boundary conditions of the microbeam imply that:

$$x=0,l:\qquad w=\frac{\partial w}{\partial x}=0 \tag{3}$$

The normalized form of the governing equation and the boundary conditions can be written as:

$$\frac{\partial^4 W}{\partial\xi^4}+\frac{\partial^2 W}{\partial\tau^2}=\left[N+\alpha\int_0^1(\frac{\partial W}{\partial\xi})^2\right]\frac{\partial^2 W}{\partial\xi^2}+\frac{V^2}{4}\left[\frac{1}{(1-W)^2}-\frac{1}{(1+W)^2}\right] \tag{4}$$

$$\xi=0,1\ :\ W=\frac{\partial W}{\partial\xi}=0 \tag{5}$$

Assuming the non-dimensional variables and parameters as:

$$\xi=\frac{x}{l}\ ,\ W=\frac{w}{g_0}\ ,\ \tau=t\sqrt{\frac{\bar{E}I}{\rho bhl^4}}\ ,\ \alpha=6\left(\frac{g_0}{h}\right)^2,\ \mathrm{N}=\frac{\bar{N}l^2}{\bar{E}I}\quad,\quad V=\sqrt{\frac{24\varepsilon_v l^4 v^2}{\bar{E}h^3 g_0^{\,3}}} \tag{6}$$

The non-dimensional deflection $W(\xi,\tau)$ is considered as:

$$W(\xi,\tau)=\sum_{i=1}^{n}\phi_i(\xi)u_i(\tau) \tag{7}$$

where $\varphi_i(\xi)$ is the $i$th eigen-function that satisfies the imposed boundary conditions, $u_i(\tau)$ is the $i$th time-dependent deflection coordinate and $n$ is the assumed degrees of freedom of the microbeam.

To solve Eq. (4), a single degree-of freedom model $(n=1)$ is considered and deflection function $W(\xi,\tau)$ is assumed to be:

$$W(\xi,\tau)=u(\tau)\phi(\xi) \tag{8}$$

Based on a common suggestion [18], the trial function is considered as $\varphi(\xi)=16\xi^2(1-\xi)^2$ in order to satisfy the boundary conditions. Equation (4) is multiplied by $(1-W^2)^2$ to avoid division by zero. Then by substituting Eq. (8) into the resulting equation and multiplying $\varphi(\xi)$ and at last integrating from 0 to 1, one can obtain [6]:

$$\ddot{u}(a_1 u^4 + a_2 u^2 + a_3) + a_4 u + a_5 u^3 + a_6 u^5 + a_7 u^7 = 0 \quad (9)$$

where

$$a_1 = \int_0^1 \phi^6 d\xi, \qquad a_2 = -2\int_0^1 \phi^4 d\xi, \qquad a_3 = \int_0^1 \phi^2 d\xi$$
$$a_4 = \int_0^1 (\phi''''\phi - N\phi''\phi - V^2\phi^2) d\xi, \qquad a_5 = \int_0^1 \left( (-2\phi''''\phi^3 + 2N\phi''\phi^3 - \alpha\phi''\phi \int_0^1 (\phi')^2 d\xi) \right) d\xi \quad (10)$$
$$a_6 = \int_0^1 (\phi''''\phi^5 - N\phi''\phi^5 + 2\alpha\phi''\phi^3 \int_0^1 (\phi')^2 d\xi) d\xi, \qquad a_7 = -\int_0^1 (\alpha\phi''\phi^5 \int_0^1 (\phi')^2 d\xi) d\xi$$

and the initial conditions are assumed as follows:

$$u(0) = A \quad , \quad \dot{u}(0) = 0$$

## 3. APPLICATION OF HE'S FREQUENCY AMPLITUDE

First Eq. (9) is considered and then the He's frequency amplitude (HFA) method is applied to solve the governing equations of the nonlinear oscillator. Considering He's frequency amplitude approach[12-14], two trial functions $u_1(t) = A\cos(\omega_1 t)$ and $u_2(t) = A\cos(\omega_2 t)$, are considered assuming $\omega_1 = 1$, $\omega_2 = \omega$ and $\omega$ is the frequency of the nonlinear oscillator. Substitution of $u_1$ and $u_2$ into Eq. (9) results in the following residuals respectively:

$$R_1(t) = (a_4 - a_3)A\cos(t) + (a_5 - a_2)A^3\cos^3(t) + (a_6 - a_1)A^5\cos^5(t) + a_7 A^7\cos(t) \quad (11)$$

and

$$R_2(t) = (a_4 - \omega^2 a_3)A\cos(\omega t) + (a_5 - \omega^2 a_2)A^3\cos^3(\omega t) + (a_6 - \omega^2 a_1)A^5\cos^5(\omega t) + a_7 A^7\cos(\omega t) \quad (12)$$

Assuming $T_1 = 2\pi$ and $T_2 = \frac{2\pi}{\omega}$, the above residuals can be expressed as weighted residuals [27]:

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1(t)\cos(\frac{2\pi}{T_1}t)dt$$
$$\tilde{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2(t)\cos(\frac{2\pi}{T_2}t)dt \quad (13)$$

Subsequently, $\omega^2$ can be calculated with a good approximation as:

$$\omega^2 = \frac{\omega_1^2\tilde{R}_2 - \omega_2^2\tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1} \quad (14)$$

Finally, the nonlinear natural frequency can be determined from Eq. (15) as:

$$\omega = \frac{\sqrt{2}}{4}\sqrt{\frac{64a_4 + 48A^2 a_5 + 40A^4 a_6 + 35A^6 a_7}{5A^4 a_1 + 6A^2 a_2 + 8a_3}} \quad (15)$$

The effects of different parameters on the nonlinear natural frequency of the system can now be investigated parametrically.

## 4. RESULTS AND DISCUSSION

In order to verify the effectiveness and the accuracy of the proposed method the results obtained here are compared with the results given by energy balance method and fourth-order Runge-Kutta method. In calculations, it is assumed that $N = 10$ and $\alpha = 24$.

Figures 2 and 3 show the results for the three methods mentioned above for two different non-dimensional exciting electrostatic forces $V = 0$ and $V = 16$ respectively. It can be seen that the results obtained by HFA are in excellent accordance with the numerical results obtained by Runge-Kutta method; even more accurate than the EBM results

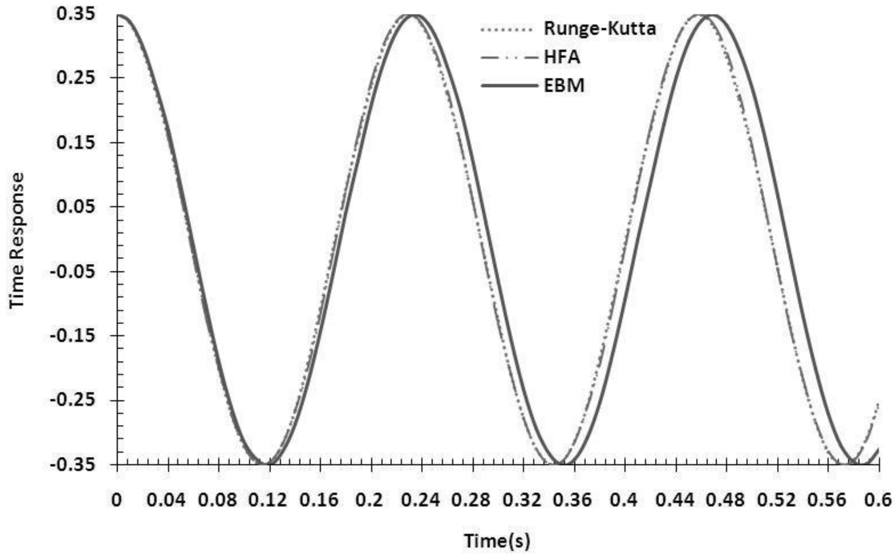


Fig. 2. Comparison between He's frequency amplitude, energy balance method [6] and the fourth-order Runge-Kutta method where *A*=0.35 and *V*=0

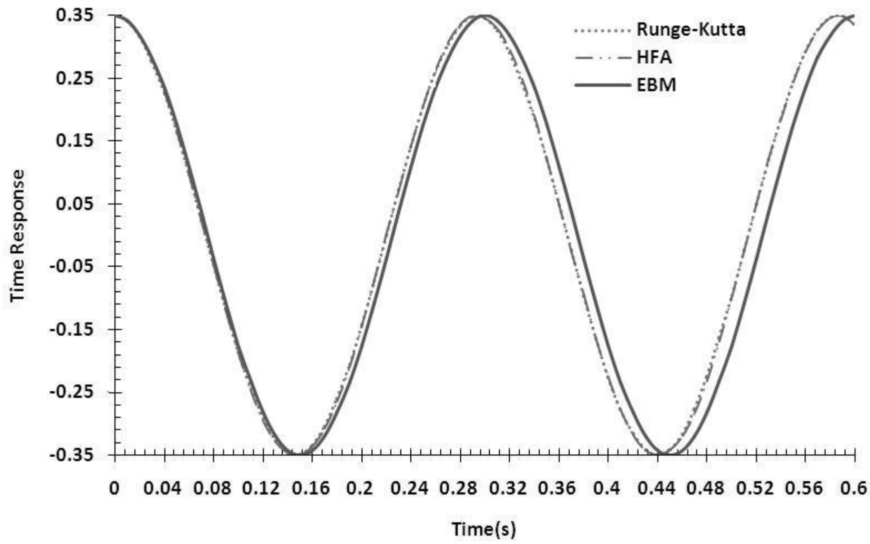


Fig. 3. Comparison between He's frequency amplitude, energy balance method [6] and the fourth-order Runge-Kutta method where *A*=0.35 and *V*=16

Figure 4 demonstrates the effects of the non-dimensional amplitude *A* and non-dimensional electrostatic load *V* on the nonlinear principal frequency of the system. As it is observable, the nonlinear principal frequency $\omega$ increases by increasing the non-dimensional amplitude *A* for different values of the non-dimensional electrostatic load *V*. In addition, it can be inferred that the nonlinear principal frequency $\omega$

decreases as the non-dimensional electrostatic load *V* increases. In other words, the voltage should be increased if a lower frequency response is desired and vice versa.

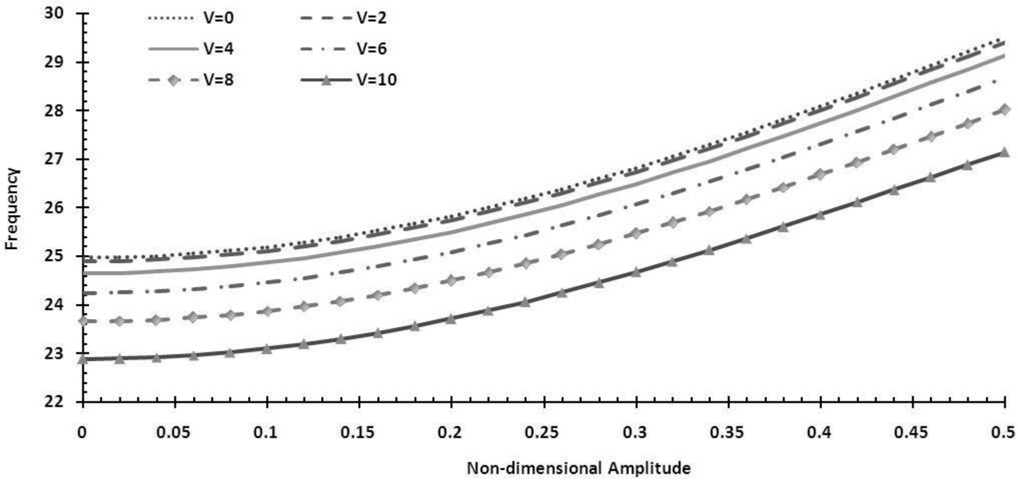


Fig. 4. Nonlinear frequency versus non-dimensional amplitude for different values of non-dimensional electrostatic load

In Figs. 5 and 6 the microbeam deflection responses for different values of the non-dimensional electrostatic load and amplitude are shown.

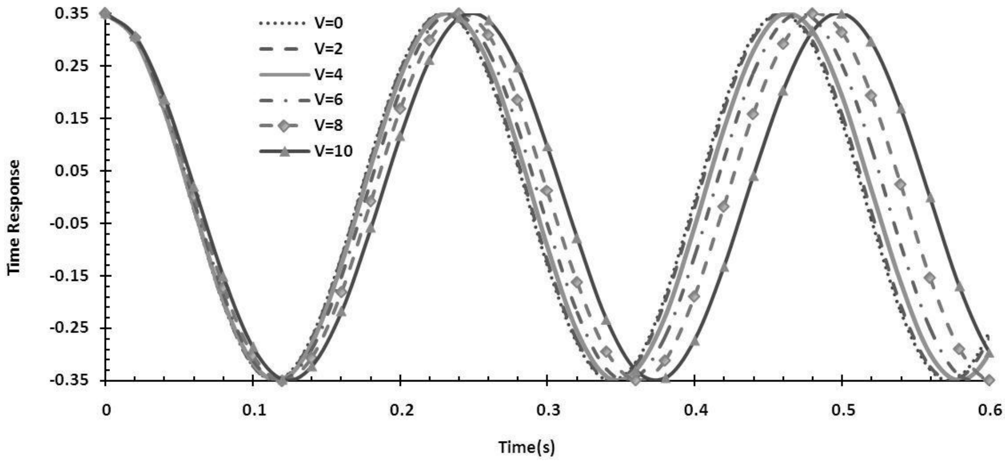


Fig. 5. Microbeam deflection for *A*=0.35 and different values of non-dimensional electrostatic load

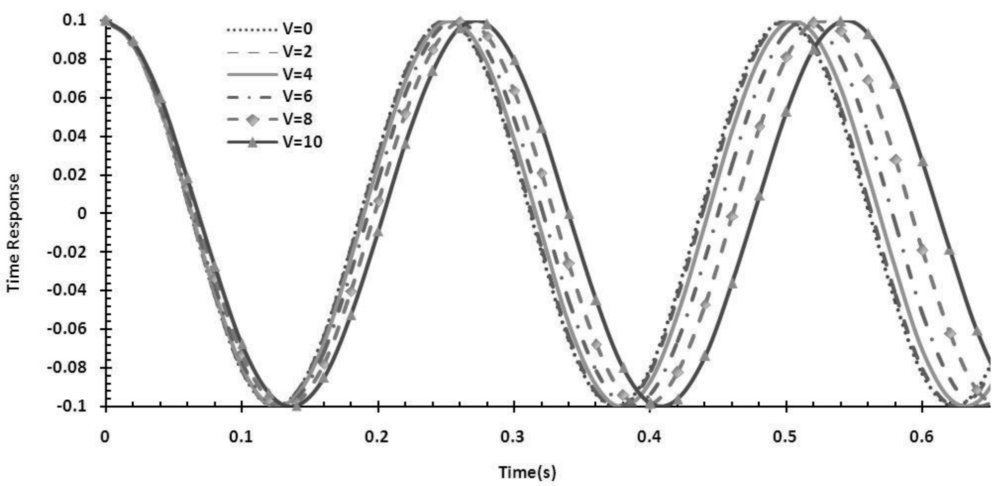


Fig. 6. Microbeam deflection for *A*=0.1 and different values of non-dimensional electrostatic load

It can be understood from Fig. 7 that an increase in both the non-dimensional amplitude and non-dimensional axial load would result in greater values of the nonlinear principal frequency. It should be noted that the non-dimensional axial loading can be either positive or negative as for tensile and compressing axial loads respectively.

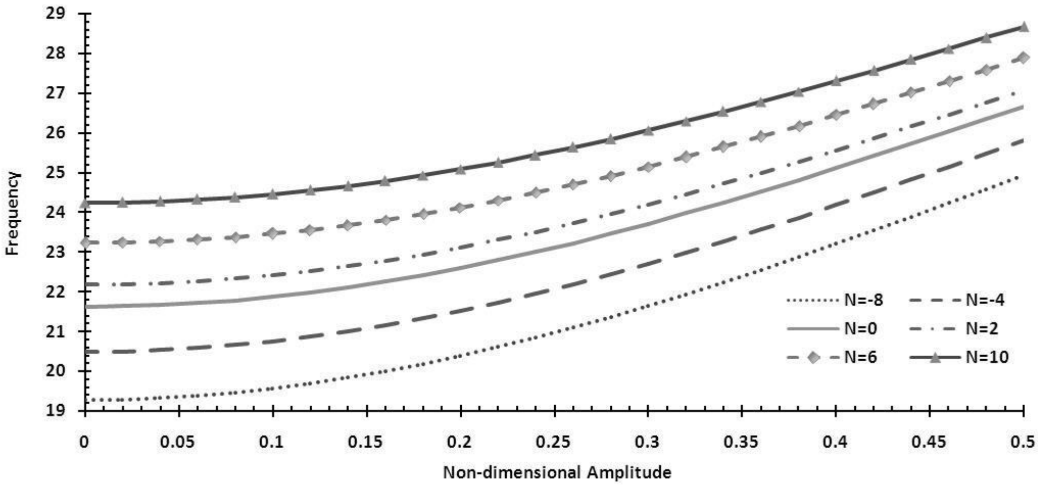


Fig. 7. Nonlinear frequency versus non-dimensional amplitude for different values of non-dimensional axial load where *V*=6

## 5. CONCLUSION

In this paper, the He's frequency amplitude formulation has been applied to investigate nonlinear oscillation of the microbeam-based micro-electro-mechanical system. The governing partial differential equation of motion derived according to the Euler-Bernoulli beam theory is transformed to a nonlinear ordinary differential equation by using the Galerkin method. The resulted equation is solved by He's frequency amplitude method and to verify the accuracy of the proposed method, the results are compared with those given by Energy Balance and fourth-order Runge-Kutta methods. The comparison indicates the accuracy, quick convergence and the effectiveness of the He's frequency amplitude approaches in solving such problems. Moreover, it is perceived that the nonlinear frequency is increased by increasing the non-dimensional amplitude, electrostatic and axial load while the other parameters are assumed to be constant.

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